Global solutions to a degenerate solutal phase-field model for the solidification of binary alloy

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Abstract

We consider a degenerate solutal phase-field model for the solidification of a binary alloy. This model is concerned with the evolution of the phase-field variable together with the relative concentration of the alloy for which the equation may degenerate. The existence of global weak solutions is proved for the degenerate case with a loss of regularity for the concentration in comparison with the non-degenerate case.

Key words : Phase-field model, degenerate parabolic systems, Faedo-Galerkin method.

1 Introduction

We investigate the existence of global weak solutions for the solutal and isothermal phase-field model of a binary alloy derived by Warren-Boettinger [8] in the case of a degenerate equation for the concentration. This degeneracy arises when the diffusion coefficient vanishes in the solid phase. This work is mainly based on a previous study done in [5] for the non-degenerate case. We also refer to [3] where a degenerate case is considered for a different phase-field model but whose part of this study has been inspired. In this paper, we are interested in a phase-field model describing the time evolution of an order parameter ϕ and the relative concentration c of the alloy. The order parameter ϕ accounts for the solidification state of the alloy and is equal to 0 in the solid phase and equals 1 in the liquid phase. The phase-field model is given by the following equations (we refer to [2],[5] for a brief description and to [8] for a full derivation) :

$$(P)\left\{\begin{array}{ll}
\dfrac{\partial\phi}{\partial t}=\varepsilon^2\Delta\phi+F_1(\phi)+cF_2(\phi) & \text{in } \Omega\times(0,+\infty),\\[2mm]
\dfrac{\partial c}{\partial t}=\operatorname{div}\Big(D_1(\phi)\big(\nabla c+D_2(c,\phi)\nabla\phi\big)\Big) & \text{in } \Omega\times(0,+\infty),\\[2mm]
\dfrac{\partial\phi}{\partial n}=\dfrac{\partial c}{\partial n}=0 & \text{on } \partial\Omega\times(0,+\infty),\\[2mm]
\phi(0)=\phi_0,\quad c(0)=c_0 & \text{in } \Omega,
\end{array}\right.$$

where Ω is a open subset of $\mathbb{R}^d$ $(d\le 3)$ with a smooth boundary $\partial\Omega$ with unit normal n, and ε is a constant. Note that, unlike the problem states in [5], the dependency of the coefficient in front of $\nabla\phi$, with the coefficient $D_1(\phi)$ is here explicitly pointed out. Indeed this will be a key ingredient for the analysis carried out through this paper.

Now let us precise the assumptions made on the non-linear terms in (P). In this work, we assume that :

(H1) $F_1,\ F_2\in C(\mathbb{R})$ are Lipschitz and bounded functions.

(H2) $D_1\in C(\mathbb{R})$ is a Lipschitz non-negative and bounded function with

$$0\le D_1(r)\le D_l,\qquad \forall\, r\in\mathbb{R}.$$

(H3) $D_2\in C(\mathbb{R}\times\mathbb{R})$ is a Lipschitz and bounded function.

Note that the degeneracy of the equation for c is allowed through assumption (H2). A typical situation is given when $D_1(\phi)$ is equal to zero for $\phi=0$ i.e. $D_1(0)=D_s=0$, corresponding to the vanishing of the diffusion coefficient in the solid case.

In addition, we also make further assumptions on the behaviour of the non-linear terms :

(H4) $F_1 \equiv F_2 \equiv 0$ in $(-\infty, 0] \cup [1, +\infty)$.

(H5) $D_2(., r) \equiv 0$ in $(-\infty, 0] \cup [1, +\infty)$ and for all $r \in \mathbb{R}$.

These assumptions lead to a maximum principle as we shall see later.

Throughout this article, we note $V = H^1(\Omega)$ and V' its dual space. The duality product between V' and V is denoted by $< . , . >_{V',V}$. Finally for $T > 0$, we put $Q_T = \Omega \times (0, T)$.

We then state the main result about the global existence of weak solutions of Problem (P) under the previous assumptions.

Theorem 1 *Let $(\phi_0, c_0) \in H^1(\Omega) \times L^2(\Omega)$ such that $\phi_0, c_0 \in [0, 1]$ a.e. in Ω. Then, under assumptions (H1)-(H5) and for any $T > 0$, there exists (ϕ, c, J) satisfying*

$$\phi \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad \phi(0) = \phi_0,$$
$$c \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; V'), \quad c(0) = c_0,$$
$$J \in L^2(Q_T)^2, \quad J = \nabla (D_1(\phi)c) - c\nabla D_1(\phi),$$

such that $\phi, c \in [0, 1]$ a.e. in Q_T and

$$\frac{\partial \phi}{\partial t} = \varepsilon^2 \Delta \phi + F_1(\phi) + cF_2(\phi) \quad a.e. \ in \ \Omega \times (0, T), \tag{1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad a.e. \ on \ \partial\Omega \times (0, T), \tag{2}$$

$$< \frac{\partial c}{\partial t}, v >_{V',V} + \int_\Omega (J + D_1(\phi) D_2(c, \phi) \nabla \phi) \cdot \nabla v \, dx = 0, \tag{3}$$

for all $v \in H^1(\Omega)$, a.e. in $(0, T)$.

The proof is based on a regularization procedure of the degenerate coefficient D_1 and passage to the limit.

2 A regularized problem

Following [3], we regularize the coefficient D_1 by introducing a continuous function $l : \mathbb{R}^+ \to \mathbb{R}$ such that $l(0) = 0$ and $l(\lambda) > 0$ for all $\lambda > 0$. Then we put

$$D_1^\lambda = D_1 + l(\lambda) \tag{4}$$

For a given $\lambda > 0$, we consider the problem (P^λ) obtained by putting the coefficient D_1^λ in place of D_1 in the front of ∇c in (P). In that way, since D_1 satisfies (H2), the regularized coefficient D_1^λ satisfies for all $\lambda \in (0, 1]$ the following property :

(H2') D_1^λ is a Lipschitz and bounded function such that $0 < l(\lambda) \leq D_1^\lambda(r) \leq L_l$, for all $r \in \mathbb{R}$.

Then an existence result holds for the problem (P^λ) that can be read as follows (see [5], Th.1) :

Let $(\phi_0, c_0) \in H^1(\Omega) \times L^2(\Omega)$ and $T > 0$. Then under assumptions (H1)-(H3), for any $\lambda \in (0, 1]$ there exists $(\phi^\lambda, c^\lambda)$ satisfying

$$\phi^\lambda \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad \phi^\lambda(0) = \phi_0, \tag{5}$$

$$c^\lambda \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V'), \quad c^\lambda(0) = c_0 \tag{6}$$

such that

$$\frac{\partial \phi^\lambda}{\partial t} = \varepsilon^2 \Delta \phi^\lambda + F_1(\phi^\lambda) + cF_2(\phi^\lambda) \qquad \text{a.e. in } Q_T, \tag{7}$$

$$\frac{\partial \phi^\lambda}{\partial n} = 0 \qquad \text{a.e. on } \partial\Omega \times (0,T), \tag{8}$$

$$< \frac{\partial c^\lambda}{\partial t}, v >_{V',V} + \int_\Omega \left(D_1^\lambda(\phi^\lambda)\nabla c^\lambda + D_1(\phi^\lambda) D_2(c^\lambda, \phi^\lambda)\nabla\phi^\lambda \right) \cdot \nabla v = 0, \tag{9}$$

for all $v \in H^1(\Omega)$, a.e. in $(0,T)$.

In addition, if we assume (H4)-(H5) then a maximum principle holds (see [5], Th.3) : If $\phi_0, c_0 \in [0,1]$ a.e. in Ω, then for all $t \in [0,T]$,

$$\phi^\lambda(t), c^\lambda(t) \in [0,1], \quad \text{a.e. in } \Omega. \tag{10}$$

Remark : Since ϕ^λ and c^λ satisfy the regularity mentioned in (5) and (6), we infer that $\phi^\lambda \in C([0,T]; H^1(\Omega))$ and $c^\lambda \in C([0,T]; L^2(\Omega))$ and thus initial conditions in (5) and (6) make sense.

3 Uniform estimates for the regularized problem

We want to pass to the limit with $\lambda \to 0$ in (7), (8) and (9). To this end, we need a priori estimates for $(\phi^\lambda, c^\lambda)$ which are uniform with respect to λ. From now, we denote by C_T a positive constant depending on T,$|\Omega|$,ε,$\|F_1\|_{L^\infty(\mathbb{R})}$, $\|F_2\|_{L^\infty(\mathbb{R})}$,$\|D_1\|_{W^{1,\infty}(\mathbb{R})}$, L_l,$\|D_2\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$ and $\|\phi_0\|_{H^1(\Omega)}$,$\|c_0\|_{L^2(\Omega)}$ but independent of λ.

Lemma 1 *There exists a positive constant C_T such that for all $\lambda \in (0,1]$, the following estimates hold :*

i) $\|\phi^\lambda\|_{L^\infty(0,T;H^1(\Omega))} + \|\phi_t^\lambda\|_{L^2(Q_T)} + \|c^\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \int_{Q_T} D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 dxdt \le C_T$

ii) $\|\phi^\lambda\|_{L^2(0,T;H^2(\Omega))} \le C_T$

iii) $\|c_t^\lambda\|_{L^2(0,T;V')} \le C_T$

Proof of Lemma 1 :
i) We take c^λ as a test function in (9). Integrating over $(0,t)$ for $t \in [0,T]$, we obtain

$$\frac{1}{2}\|c^\lambda(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 = \frac{1}{2}\|c_0\|^2_{L^2(\Omega)} - \int_0^t\int_\Omega D_1(\phi^\lambda)D_2(c^\lambda,\phi^\lambda)\nabla\phi^\lambda \cdot \nabla c^\lambda. \tag{11}$$

Using the boundedness of D_2, the positivity of D_1 in assumptions (H2),(H3) and the definition (4) of D_1^λ together with Cauchy-Schwarz and Young inequalities, we infer that, for all $t \in [0,T]$,

$$\left| \int_0^t\int_\Omega D_1(\phi^\lambda)D_2(c^\lambda,\phi^\lambda)\nabla\phi^\lambda \cdot \nabla c^\lambda \right| \le \frac{1}{2}\int_0^t\int_\Omega D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 + \frac{C_T}{2}\int_0^t\int_\Omega |\nabla\phi^\lambda|^2. \tag{12}$$

Then combining (11) with (12), we deduce that there exists a positive constant C_1 independent of λ such that, for all $t \in [0,T]$,

$$\|c^\lambda(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 \le C_1\left(1 + \int_0^t\int_\Omega |\nabla\phi^\lambda|^2\right). \tag{13}$$

Now we take the L^2-scalar product of (7) with $\phi_t^\lambda + \phi^\lambda$ and we integrate over $(0,t)$ for $t \in [0,T]$. After integration by parts we obtain, for all $t \in [0,T]$,

$$\begin{aligned} \frac{1}{2}\left(\|\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)}\right) &+ \int_0^t \left(\|\phi_t^\lambda\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda\|^2_{L^2(\Omega)}\right) \\ = \frac{1}{2}&\left(\|\phi_0\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi_0\|^2_{L^2(\Omega)}\right) \\ &+ \int_0^t\int_\Omega F_1(\phi^\lambda)(\phi_t^\lambda + \phi^\lambda) + \int_0^t\int_\Omega c^\lambda F_2(\phi^\lambda)(\phi_t^\lambda + \phi^\lambda). \end{aligned} \tag{14}$$

Then we use Cauchy-Schwarz and Young inequalities with (H1) to estimate the right-hand term in (14). This leads to the existence of a positive constant C_2 independent of λ such that

$$\|\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)} + \int_0^t \left(\|\phi^\lambda_t\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda\|^2_{L^2(\Omega)}\right) \le C_2\left(1 + \int_0^t \|\phi^\lambda\|^2_{L^2(\Omega)} + \int_0^t \|c^\lambda\|^2_{L^2(\Omega)}\right) \tag{15}$$

Now we multiply estimate (13) by a number $\delta > 0$ that will be choosen later and adding the result to estimate (15), we get, for all $t \in [0,T]$,

$$\|\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)} + \delta\|c^\lambda(t)\|^2 + \int_0^t \|\phi^\lambda_t\|^2_{L^2(\Omega)} + \varepsilon^2\int_0^t \|\nabla\phi^\lambda\|^2_{L^2(\Omega)}$$
$$+ \delta\int_0^t\int_\Omega D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 \ \le\ \delta C_1\int_0^t \|\nabla\phi^\lambda\|^2_{L^2(\Omega)} + C_T\left(1 + \int_0^t \|\phi^\lambda\|^2_{L^2(\Omega)} + \int_0^t \|c^\lambda\|^2_{L^2(\Omega)}\right).$$

We choose $\delta > 0$ such that $\delta C_1 = \varepsilon^2/2$ and thus we obtain for all $t \in [0,T]$,

$$\|\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)} + \delta\|c^\lambda(t)\|^2 + \int_0^t \|\phi^\lambda_t\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{2}\int_0^t \|\nabla\phi^\lambda\|^2_{L^2(\Omega)} + \delta\int_0^t\int_\Omega D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 \ \le\ C_T\left(1 + \int_0^t \|\phi^\lambda\|^2_{L^2(\Omega)} + \int_0^t \|c^\lambda\|^2_{L^2(\Omega)}\right). \tag{16}$$

Applying Gronwall's lemma, we conclude that

$$\|\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)} + \delta\|c^\lambda(t)\|^2 \le C_T, \qquad \text{for all } t \in [0,T]$$

and thus

$$\|\phi^\lambda\|_{L^\infty(0,T;H^1(\Omega))} + \|c^\lambda\|_{L^\infty(0,T;L^2(\Omega))} \le C_T. \tag{17}$$

Finally, estimate (16) with (17) leads to the desired estimate *i)* of Lemma 1.

ii) We take the L^2-scalar product of equation (7) with $-\Delta\phi^\lambda$ and we integrate over $(0,t)$ for $t \in [0,T]$. After integration by parts and the use of Cauchy-Schwarz and Young inequalities with (H1), we have, for all $t \in [0,T]$,

$$\|\nabla\phi^\lambda(t)\|^2_{L^2(\Omega)} + \varepsilon^2\int_0^t \|\Delta\phi^\lambda\|^2_{L^2(\Omega)} \le C_T\left(1 + \int_0^t \|c^\lambda\|^2_{L^2(\Omega)}\right). \tag{18}$$

We deduce using estimate *i)* of Lemma 1 that

$$\|\Delta\phi^\lambda\|_{L^2(Q_T)} \le C_T. \tag{19}$$

We conclude using well-known elliptic results that estimate *ii)* holds.

iii) From equation (9) with (H1),(H2),(H3), it is straightforward to get

$$\|c^\lambda_t\|_{L^2(0,T;V')} \le C_T\left(\int_{Q_T} D_1^\lambda(\phi^\lambda)|\nabla c^\lambda|^2 + \|\phi^\lambda\|^2_{L^\infty(0,T;H^1(\Omega))}\right). \tag{20}$$

Then we conclude using estimate *i)*, that estimate *iii)* holds. □

4 Passage to the limit with $\lambda \to 0$.

We want now to pass to the limit in (7), (8) and (9) as λ tends to 0. From Lemma 1, we know that $\{\phi^\lambda\}$ is bounded in

$$W_1 = \{v \in L^2(0,T;H^2(\Omega)),\ v_t \in L^2(0,T;L^2(\Omega))\}$$

and in

$$W_2 = \{v \in L^\infty(0,T;H^1(\Omega)),\ v_t \in L^2(0,T;L^2(\Omega))\}.$$

Since W_1 and W_2 are compactly embedded respectively into $L^2(0,T;H^1(\Omega))$ and into $C([0,T];L^2(\Omega))$ (see [6], Cor. 4), then there exists a function

$$\phi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$$

and a subsequence still denoted by ϕ^λ, such that as λ tends to 0,

$$\phi^\lambda \rightharpoonup \phi \text{ in } L^2(0,T;H^2(\Omega)) \text{ weak,} \tag{21}$$
$$\phi^\lambda_t \rightharpoonup \phi_t \text{ in } L^2(Q_T) \text{ weak,} \tag{22}$$
$$\phi^\lambda \to \phi \text{ in } L^2(0,T;H^1(\Omega)) \text{ and in } C([0,T];L^2(\Omega)) \text{ strong.} \tag{23}$$

On the other hand, we know that $\{c^\lambda\}$ is bounded in

$$W_3 = \{v \in L^\infty(0,T;L^2(\Omega)),\ v_t \in L^2(0,T;V')\}$$

and in $L^\infty(Q_T)$ due to the maximum principle. Since W_3 is compactly embedded in $C([0,T];V')$ (see [6]), there exists a function

$$c \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;V') \cap L^\infty(Q_T)$$

and a subsequence still denoted by c^λ, such that as λ tends to 0,

$$c^\lambda \rightharpoonup c \text{ in } L^2(Q_T) \text{ weak, and in } L^\infty(Q_T) \text{ weak-*,} \tag{24}$$
$$c^\lambda_t \rightharpoonup c_t \text{ in } L^2(0,T;V') \text{ weak,} \tag{25}$$
$$c^\lambda \to c \text{ in } C([0,T];V') \text{ strong.} \tag{26}$$

Finally, from Lemma 1-*i)* and property (H2'), we infer that $D_1^\lambda(\phi^\lambda)\nabla c^\lambda$ is uniformly bounded in $L^2(Q_T)^2$ and thus there exists a function

$$J \in L^2(Q_T)^2 \tag{27}$$

such that

$$D_1^\lambda(\phi^\lambda)\nabla c^\lambda \rightharpoonup J \text{ in } L^2(Q_T)^2 \text{ weak .} \tag{28}$$

We now obtain a first convergence result for the nonlinear terms.

Lemma 2 *As $\lambda \to 0$, we have*

i)	$F_i(\phi^\lambda) \to F_i(\phi)$ $\quad i = 1,2$	*in* $L^p(Q_T)$, $\forall p \in [1,+\infty)$,	(29)
ii)	$c^\lambda F_2(\phi^\lambda) \rightharpoonup cF_2(\phi)$	*in* $L^2(Q_T)$ *weak* ,	(30)
iii)	$D_1(\phi^\lambda) \to D_1(\phi)$	*in* $L^2(0,T;H^1(\Omega))$,	(31)
iv)	$c^\lambda D_1(\phi^\lambda) \rightharpoonup cD_1(\phi)$	*in* $L^2(Q_T)$ *weak,*	(32)
v)	$c^\lambda \nabla D_1(\phi^\lambda) \rightharpoonup c\nabla D_1(\phi)$	*in* $L^2(Q_T)^2$ *weak.*	(33)

Proof of Lemma 2 :
i) From (23), we have that $\phi^\lambda \to \phi$ a.e. in Q_T and since F_i are Lipschitz functions, we conclude by Lebesgue dominated convergence theorem that $F_i(\phi^\lambda) \to F_i(\phi)$ in $L^p(Q_T)$ for all $p \in [1,+\infty)$.

ii) For $v \in L^2(Q_T)$, we have

$$\int_{Q_T} \left(c^\lambda F_2(\phi^\lambda) - cF_2(\phi)\right) v = \int_{Q_T} (c^\lambda - c)F_2(\phi)v + \int_{Q_T} c^\lambda \left(F_2(\phi^\lambda) - F_2(\phi)\right) v.$$

The first term on the right hand side tends to 0 since $F_2(\phi)v$ belongs to $L^2(Q_T)$ and due to (24). The second term also tends to 0 since $0 \le c^\lambda \le 1$ a.e. in Q_T and thanks to (29).

iii) D_1 is Lipschitz and thanks to (23), we conclude (see [1], Th. 16.7) that $D_1(\phi^\lambda) \to D_1(\phi)$ in $L^2(0,T;H^1(\Omega))$.

iv) The proof is the same as for *ii)*.

v) Let $v \in L^2(Q_T)^2$. We have

$$\int_{Q_T} \left(c^\lambda \nabla D_1(\phi^\lambda) - c\nabla D_1(\phi)\right) \cdot v = \int_{Q_T} (c^\lambda - c)\nabla D_1(\phi) \cdot v + \int_{Q_T} c^\lambda \left(\nabla D_1(\phi^\lambda) - \nabla D_1(\phi)\right) \cdot v.$$

The first term on the right hand side tends to 0 since $D_1 \in W^{1,\infty}(\mathbb{R})$ and $\nabla D_1(\phi)\cdot v = D_1'(\phi)\nabla\phi\cdot v \in L^2(Q_T)$ and due to (24). The second term tends also to 0 since *iii)* holds and $0 \leq c^\lambda \leq 1$ a.e. on Q_T. Hence $c^\lambda \nabla D_1(\phi^\lambda) \rightharpoonup c\nabla D_1(\phi)$ in $L^2(Q_T)^2$ weak.

□

We can now identify the function J as it is done in [3].

Lemma 3 *As $\lambda \to 0$, we have*

$$D_1^\lambda(\phi^\lambda)\nabla c^\lambda \rightharpoonup J \text{ in } L^2(Q_T)^2 \text{ weak.} \tag{34}$$

with

$$J = \nabla\left(D_1(\phi)c\right) - c\nabla D_1(\phi) \tag{35}$$

Proof of Lemma 3 : The proof is similar to [3]. Since $D_1(\phi^\lambda) \in L^2(0,T;H^1(\Omega))$ and $c^\lambda \in L^\infty(0,T;H^1(\Omega))$, we first infer that $D_1(\phi^\lambda)c^\lambda \in L^2(0,T;W^{1,3/2}(\Omega))$ (see for instance [7]) and

$$\nabla\left(D_1(\phi^\lambda)c^\lambda\right) = D_1(\phi^\lambda)\nabla c^\lambda + c^\lambda\nabla D_1(\phi^\lambda) \quad \text{a.e. in } Q_T. \tag{36}$$

Furthermore, since $0 \leq c^\lambda \leq 1$ a.e. in Q_T, we infer that $D_1(\phi^\lambda)c^\lambda \in L^2(0,T;H^1(\Omega))$.
Now, from Lemma 2-*iv)*, we deduce that

$$\nabla\left(D_1(\phi^\lambda)c^\lambda\right) \to \nabla\left(D_1(\phi)c\right) \text{ in } \mathcal{D}'(Q_T)$$

and from Lemma 2-*v)* that

$$c^\lambda\nabla D_1(\phi^\lambda) \to c\nabla D_1(\phi) \text{ in } \mathcal{D}'(Q_T).$$

Thus (36) leads to

$$D_1(\phi^\lambda)\nabla c^\lambda \to \nabla\left(D_1(\phi)c\right) - c\nabla D_1(\phi) \text{ in } \mathcal{D}'(Q_T). \tag{37}$$

Moreover, recall that $D_1^\lambda = D_1 + l(\lambda)$ and then since D_1 is nonnegative, we deduce from Lemma 1-*i)*,

$$l(\lambda)\int_{Q_T} |\nabla c^\lambda|^2 \leq C_T.$$

Therefore $\|l(\lambda)\nabla c^\lambda\|^2_{L^2(Q_T)} \leq C_T l(\lambda)$ and then $l(\lambda)\nabla c^\lambda \to 0$ in $L^2(Q_T)^2$ as $\lambda \to 0$. Hence we deduce from (37) that

$$D_1^\lambda(\phi^\lambda)\nabla c^\lambda \to \nabla\left(D_1(\phi)c\right) - c\nabla D_1(\phi) \text{ in } \mathcal{D}'(Q_T). \tag{38}$$

We conclude from (28) and (38) that $J = \nabla\left(D_1(\phi)c\right) - c\nabla D_1(\phi)$.

□

The next Lemma states a further strong convergence result about the function product $D_1(\phi^\lambda)c^\lambda$ and allows us to pass to the limit in the term involving with D_2.

Lemma 4 *i) There exists a positive constant C_T such that for all $\lambda \in (0,1]$, we have*

$$\|D_1(\phi^\lambda)c^\lambda\|_{L^2(0,T;H^1(\Omega))} \leq C_T. \tag{39}$$

ii) As $\lambda \to 0$, we have

$$a) \quad D_1(\phi^\lambda)c^\lambda \to D_1(\phi)c \quad \text{in } L^2(Q_T), \tag{40}$$

$$b) \quad D_1(\phi^\lambda)D_2(c^\lambda,\phi^\lambda)\nabla\phi^\lambda \rightharpoonup D_1(\phi)D_2(c,\phi)\nabla\phi \quad \text{in } L^1(Q_T)^2 \text{ weak.} \tag{41}$$

Proof of Lemma 4 :
i) We prove that $D_1(\phi^\lambda)c^\lambda$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. From (36), we deduce that

$$\int_{Q_T} |\nabla \left(D_1(\phi^\lambda)c^\lambda\right)|^2 \leq \int_{Q_T} D_1(\phi^\lambda)^2|\nabla c^\lambda|^2 + \int_{Q_T} |c^\lambda D_1'(\phi^\lambda)\nabla\phi^\lambda|^2.$$

Then using Lemma 1-*i)* with the fact that $D_1 \leq D_1^\lambda$ and $D_1 \in W^{1,\infty}(\mathbb{R})$, we obtain

$$\begin{aligned}\int_{Q_T} |\nabla \left(D_1(\phi^\lambda)c^\lambda\right)|^2 &\leq C_T\left(1+\int_{Q_T}|c^\lambda\nabla\phi^\lambda|^2\right)\\ &\leq C_T\left(1+\|\nabla\phi^\lambda\|^2_{L^2(Q_T)}\right)\\ &\leq C_T,\end{aligned} \tag{42}$$

where we have also used the fact that $0 \leq c^\lambda \leq 1$ a.e. in Q_T and estimate *i)* of Lemma 1.

ii)-a) Since $D_1(\phi^\lambda(t))c^\lambda(t) - D_1(\phi(t))c(t) \in H^1(\Omega)$ for a.e. $t \in (0,T)$, we can apply a well-known compacteness inequality (see for instance [4], Lemma 5.1) : For all $\eta > 0$, there exists a positive constant C_η such that

$$\begin{aligned}\|D_1(\phi^\lambda(t))c^\lambda(t) - D_1(\phi(t))c(t)\|_{L^2(\Omega)} &\leq \eta\|D_1(\phi^\lambda(t))c^\lambda(t) - D_1(\phi(t))c(t)\|_{H^1(\Omega)}\\ &\quad +C_\eta\|D_1(\phi^\lambda(t))c^\lambda(t) - D_1(\phi(t))c(t)\|_{V'}.\end{aligned}$$

Then integrating over $t \in (0,T)$, we have : $\forall\eta > 0$, there exists C_η such that

$$\begin{aligned}\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(Q_T)} &\leq \eta\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;H^1(\Omega))}\\ &\quad +C_\eta\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;V')}.\end{aligned} \tag{43}$$

Moreover, we have

$$\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;V')} \leq \|D_1(\phi)(c^\lambda - c)\|_{L^2(0,T;V')} + \|c^\lambda(D_1(\phi^\lambda) - D_1(\phi))\|_{L^2(0,T;V')}$$

and since D_1 is a bounded function and $0 \leq c^\lambda \leq 1$ a.e. in Q_T, there exists a constant C_1 independent of λ such that

$$\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;V')} \leq C_1\left(\|c^\lambda - c\|_{L^2(0,T;V')} + \|D_1(\phi^\lambda) - D_1(\phi)\|_{L^2(0,T;V')}\right). \tag{44}$$

Combining (43) with (44), we obtain : $\forall\eta > 0$, there exists C'_η such that

$$\begin{aligned}\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(Q_T)} &\leq \eta\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;H^1(\Omega))}\\ &\quad +C'_\eta\left(\|c^\lambda - c\|_{L^2(0,T;V')} + \|D_1(\phi^\lambda) - D_1(\phi)\|_{L^2(0,T;V')}\right).\end{aligned}$$

Now let $\varepsilon > 0$ be fixed. Thanks to (39), there exists a positive constant C_2 independent of λ such that $\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(0,T;H^1(\Omega))} \leq C_2$ and then

$$\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(Q_T)} \leq \frac{\varepsilon}{2} + C'_\eta\left(\|c^\lambda - c\|_{L^2(0,T;V')} + \|D_1(\phi^\lambda) - D_1(\phi)\|_{L^2(0,T;V')}\right),$$

if we choose η such that $\eta C_2 \leq \varepsilon/2$. We conclude using the strong convergences (26) and (31) that

$$\|D_1(\phi^\lambda)c^\lambda - D_1(\phi)c\|_{L^2(Q_T)} \leq \varepsilon \quad \text{for } \lambda \text{ small enough},$$

and therefore $D_1(\phi^\lambda)c^\lambda \to D_1(\phi)c$ in $L^2(Q_T)$ as $\lambda \to 0$.

ii)-b) Let $v \in L^\infty(Q_T)^2$ and we put

$$\begin{aligned}\mathcal{A} &= \int_{Q_T}\left(D_1(\phi^\lambda)D_2(c^\lambda,\phi^\lambda)\nabla\phi^\lambda - D_1(\phi)D_2(c,\phi)\nabla\phi\right)\cdot v\\ &= \int_{Q_T} D_1(\phi^\lambda)\left(D_2(c^\lambda,\phi^\lambda) - D_2(c,\phi)\right)\nabla\phi\cdot v \qquad (E1)\\ &\quad +\int_{Q_T}\left(D_1(\phi^\lambda) - D_1(\phi)\right)D_2(c,\phi)\nabla\phi\cdot v \qquad (E2)\\ &\quad +\int_{Q_T} D_1(\phi^\lambda)D_2(c^\lambda,\phi^\lambda)\left(\nabla\phi^\lambda - \nabla\phi\right)\cdot v \qquad (E3)\end{aligned}$$

From (23) and Lemma 2-*iii)*, we obtain $E2 \to 0$ and $E3 \to 0$. Using the Lipschitz property of D_2, we get

$$|E1| \leq C_T \int_{Q_T} D_1(\phi^\lambda) \left(|c^\lambda - c| + |\phi^\lambda - \phi|\right) |\nabla\phi|. \tag{45}$$

Since D_1 is a nonnegative function, we can write

$$\int_{Q_T} D_1(\phi^\lambda) \left(|c^\lambda - c| + |\phi^\lambda - \phi|\right) |\nabla\phi| = \int_{Q_T} |D_1(\phi^\lambda)c^\lambda - D_1(\phi^\lambda)c||\nabla\phi| + \int_{Q_T} D_1(\phi^\lambda)|\phi^\lambda - \phi||\nabla\phi|$$

and then with (45) this leads to

$$\begin{aligned} |E1| \quad \leq \quad C_T \Bigg(& \int_{Q_T} |D_1(\phi^\lambda)c^\lambda - D_1(\phi)c||\nabla\phi| + \int_{Q_T} |D_1(\phi) - D_1(\phi^\lambda)||c||\nabla\phi| \\ & + \int_{Q_T} |\phi^\lambda - \phi||\nabla\phi| \Bigg). \end{aligned}$$

From the strong convergences (40),(23) and Lemma 2 with the fact that $c \in L^\infty(Q_T)$, we conclude that $E1 \to 0$. Thus we have proved that $\mathcal{A} \to 0$.

□

Remark : Without using the maximum principle i.e. the fact that $0 \leq c^\lambda \leq 1$ in Q_T, we can obtain weaker convergence results in Lemma 2 and Lemma 3 but which are still enough to pass to the limit in the related nonlinear terms. However, the L^∞-bound for c^λ is necessary in order to get estimate (39).

Now we are able to pass to the limit with $\lambda \to 0$ in equations (7), (8) and (9) by the use of Lemma 2, 3 and 4 and we obtain equations (1), (2) and (3). In addition, from (23) and (26) we infer that $\phi(0) = \phi_0$ and $c(0) = c_0$. Finally the weak convergence (24) leads to $\|c\|_{L^\infty(Q_T)} \leq \liminf \|c^\lambda\|_{L^\infty(Q_T)}$ which can be used to prove that $0 \leq c \leq 1$ a.e. in Q_T. The same holds for the function ϕ and the proof of Theorem 1 is completed.

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